Construction of Koszul algebras by finite Galois covering *

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Abstract

It is shown that, the quasi-Koszulities of algebras and modules are Morita invariance. A finite-dimensional K-algebra A with an action of G is quasi-Koszul if and only if so is the skew group algebra $A * G$, where G is a finite group satisfying char$K \nmid |G|$. A finite-dimensional G-graded K-algebra A is quasi-Koszul if and only if so is the smash product $A\#G^*$, where G is a finite group satisfying char$K \nmid |G|$. These results are applied to prove that, if a finite-dimensional connected quiver algebra is Koszul then so are its Galois covering algebras with finite Galois group G satisfying char$K \nmid |G|$. So one can construct Koszul algebras by finite Galois covering. Moreover, a general construction of Koszul algebras by Galois covering with finite cyclic Galois group is provided. As examples, many Koszul algebras are constructed from exterior algebras and Koszul preprojective algebras by finite Galois covering with either cyclic or noncyclic Galois group.

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Introduction

Koszul algebras play an important role in commutative algebra, algebraic geometry, algebraic topology, Lie theory and quantum groups (cf. [29, 4, 20, 6, 8, 33, 7]). It is a quite nice class of algebras because, on one hand, there exists Koszul duality in the sense of not only algebra (cf. [7, Theorem 2.10.2]

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and [18, Theorem 2.3]) but also module category and derived category (cf. [18, Theorem 5.2] and [7, Theorem 2.12.6]), on the other hand, for a Koszul algebra, both the minimal projective resolution of its semisimple part and its minimal projective bimodule resolution are rather clear (cf. [10] and [16]). Now many algebras are known to be Koszul, such as path algebras [18, p. 240], radical square zero algebras [23, Corollary], quadratic global dimension 2 algebras [17, Theorem 7.2], finite-dimensional indecomposable radical cube zero selfinjective algebras of infinite representation type [23, Theorem 1.5] and many (generalized) preprojective algebras (cf. [15, Section 7] and [23, Theorem 1.9]). From the known Koszul algebras, there are some ways to construct new Koszul algebras: the opposite algebra of a Koszul algebra is Koszul (cf. [7, Proposition 2.2.1] and [18, Corollary 4.3]), the quadratic duality equivalently the Yoneda algebra of a Koszul algebra is Koszul (cf. [7, Proposition 2.9.1], [17, Theorem 6.1] and [18, Theorem 2.2]), the tensor product algebra of two Koszul algebras is Koszul [18, Theorem 3.7], and so on.

Here we provide another construction of Koszul algebras by finite Galois covering. This construction is very convenient in practice because one can construct Koszul algebras and provide their quivers and relations directly from a known Koszul algebra given by quiver with relations. The paper is organized as follows: In Section 1, we show that the quasi-Koszulities of algebras and modules are Morita invariance. Section 2 is essentially due to Martínez-Villa (cf. [25]). In this section, we show that a finite-dimensional K-algebra A with an action of G is quasi-Koszul if and only if so is the skew group algebra $A * G$, where G is a finite group satisfying char$K \nmid |G|$. Moreover, we prove that the Yoneda algebra $\mathrm{Ext}^*_{A*G}((A*G)/J(A*G),(A*G)/J(A*G))$ of the skew group algebra $A*G$ is isomorphic to the skew group algebra $\mathrm{Ext}^*_A(A/J,A/J)*G$ of the Yoneda algebra $\mathrm{Ext}^*_A(A/J,A/J)$. In Section 3, we show in two ways that a finite-dimensional G-graded K-algebra A is quasi-Koszul if and only if so is the smash product $A\#G^*$, where G is a finite group satisfying char$K \nmid |G|$. Moreover, we prove that the Yoneda algebra $\mathrm{Ext}^*_{A\#G^*}((A\#G^*)/J(A\#G^*),(A\#G^*)/J(A\#G^*))$ of the smash product $A\#G^*$ is isomorphic to the smash product $\mathrm{Ext}^*_A(A/J,A/J)\#G^*$ of the Yoneda algebra $\mathrm{Ext}^*_A(A/J,A/J)$. In Section 4, we apply the results above to show that, if a finite-dimensional connected quiver algebra is Koszul then so are its Galois covering algebras with finite Galois group G satisfying char$K \nmid |G|$. Finally, in Section 5, we provide a general construction of Koszul algebras by Galois covering with finite cyclic Galois group. As examples, we construct many Koszul algebras from exterior algebras and Koszul preprojective algebras by finite Galois covering with either cyclic or noncyclic Galois group.

1 Morita invariance of quasi-Koszulity

In this section we show that the quasi-Koszulities of algebras and modules are Morita invariance. Throughout the paper, the composition of maps is written from right to left except for those in Yoneda algebras.

1.1 Quasi-Koszulity

Let K be a field. A graded K-algebra $A = \coprod_{i\geq 0} A_i$ is said to be *generated in degree 0 and 1* if $A_i = A_1^i$ for all $i \geq 2$ (cf. [18]).

In this section, A is assumed to be a fixed noetherian semiperfect K-algebra (cf. [1, §27]). Denote by $J := J(A)$ the Jacobson radical of A. Let M, N be two A-modules. Then $\mathrm{Ext}_A^i(M, N)$, $i \geq 1$, can be viewed as a K-vector space of congruence classes of i-extensions whose addition is given by Baer sum (cf. [21, Chapter III, §5]). It is convenient to write an i-extension in the form $M \twoheadleftarrow E_1 \leftarrow \cdots \leftarrow E_i \hookleftarrow N$. It is well-known that $\mathrm{Ext}_A^*(M, M) := \coprod_{i\geq 0} \mathrm{Ext}_A^i(M, M)$ is a graded K-algebra with $\mathrm{Ext}_A^i(M, M)$ in degree i, for which the multiplication is given by the Yoneda product (cf. [21, Chapter III, §5]). The algebra A is called a *quasi-Koszul algebra* if $\mathrm{Ext}_A^*(A/J, A/J)$ is generated in degree 0 and 1 (cf. [17, p. 263]). An A-module M is said to be *quasi-Koszul* if $\mathrm{Ext}_A^i(M, A/J)\mathrm{Ext}_A^1(A/J, A/J) = \mathrm{Ext}_A^{i+1}(M, A/J)$ for all $i \geq 0$, equivalently, $\mathrm{Ext}_A^i(M, A/J)\mathrm{Ext}_A^j(A/J, A/J) = \mathrm{Ext}_A^{i+j}(M, A/J)$ for all $i, j \geq 0$.

A graded K-algebra $A = \coprod_{i\geq 0} A_i$ is called a *graded quiver algebra* if it satisfies: (1): $A_0 \cong K^r$ as K-algebras for some $r \geq 1$; (2): $\dim_K A_i < \infty$ for all $i \geq 0$; (3): A is generated in degree 0 and 1 (cf. [24]). Note that a graded quiver algebra A is isomorphic to a graded quotient of a path algebra with length grading, namely $A \cong KQ/I$ where Q is a finite quiver and $I \subseteq (KQ^+)^2$ is a homogeneous ideal of A in the length grading, here KQ^+ denotes the ideal of KQ generated by all arrows (cf. [15]). For the theory of quivers and their representations we refer to [3]. A quasi-Koszul graded quiver algebra is called a *Koszul algebra*.

1.2 Decomposition of extension groups

Let $\oplus_{i=1}^s M_i$ be the direct sum of A-modules $M_1, ..., M_s$. Denote by $\lambda_i : M_i \to \oplus_{i=1}^s M_i, m_i \mapsto (0, ..., 0, m_i, 0, ..., 0)$ and $\rho_i : \oplus_{i=1}^s M_i \to M_i, (m_1, ..., m_s) \mapsto m_i$ the canonical injection and projection respectively. Denote by $\Delta : M \to M^s, m \mapsto (m, ..., m)$ and $\nabla : M^s \to M, (m_1, ..., m_s) \mapsto m_1 + \cdots + m_s$ the *diagonal map* and the *sum map* respectively. Note that $\Delta = \sum_{i=1}^s \lambda_i$ and $\nabla = \sum_{i=1}^s \rho_i$.

For $r \geq 0$, $\mathrm{Ext}_A^r(\oplus_{i=1}^s M_i, N) \cong \oplus_{i=1}^s \mathrm{Ext}_A^r(M_i, N)$ as K-vector spaces and the isomorphism is given by $\phi : \mathrm{Ext}_A^r(\oplus_{i=1}^s M_i, N) \to \oplus_{i=1}^s \mathrm{Ext}_A^r(M_i, N)$, $[\xi] \mapsto (\mathrm{Ext}_A^r(\lambda_i, N)([\xi]))_i$ with inverse $\psi : \oplus_{i=1}^s \mathrm{Ext}_A^r(M_i, N) \to \mathrm{Ext}_A^r(\oplus_{i=1}^s M_i, N)$, $([\xi_i])_i \mapsto \sum_{i=1}^s \mathrm{Ext}_A^r(\rho_i, N)([\xi_i])$.

Similarly, $\mathrm{Ext}_A^r(M, \oplus_{j=1}^t N_j) \cong \oplus_{j=1}^t \mathrm{Ext}_A^r(M, N_j)$ as K-vector spaces and the isomorphism is given by $\phi : \mathrm{Ext}_A^r(M, \oplus_{j=1}^t N_j) \to \oplus_{j=1}^t \mathrm{Ext}_A^r(M, N_j)$, $[\xi] \mapsto (\mathrm{Ext}_A^r(M, \rho_j)([\xi]))_j$ with inverse $\psi : \oplus_{j=1}^t \mathrm{Ext}_A^r(M, N_j) \to \mathrm{Ext}_A^r(M, \oplus_{j=1}^t N_j)$, $([\xi_j])_j \mapsto \sum_{i=1}^t \mathrm{Ext}_A^r(M, \lambda_j)([\xi_j])$.

More general, $\mathrm{Ext}_A^r(\oplus_{i=1}^s M_i, \oplus_{j=1}^t N_j) \cong \oplus_{i=1}^s \oplus_{j=1}^t \mathrm{Ext}_A^r(M_i, N_j)$ as K-vector spaces and the isomorphism is given by $\phi : \mathrm{Ext}_A^r(\oplus_{i=1}^s M_i, \oplus_{j=1}^t N_j) \to \oplus_{i=1}^s \oplus_{j=1}^t \mathrm{Ext}_A^r(M_i, N_j)$, $[\xi] \mapsto (\mathrm{Ext}_A^r(M_i, \rho_j)\mathrm{Ext}_A^r(\lambda_i, \oplus_{j=1}^t N_j)([\xi]))_{i,j}$ with inverse $\psi : \oplus_{i=1}^s \oplus_{j=1}^t \mathrm{Ext}_A^r(M_i, N_j) \to \mathrm{Ext}_A^r(\oplus_{i=1}^s M_i, \oplus_{j=1}^t N_j)$, $([\xi_{ij}])_{i,j} \mapsto \sum_{i=1}^s \sum_{j=1}^t \mathrm{Ext}_A^r(\rho_i, \oplus_{j=1}^t N_j)\mathrm{Ext}_A^r(M_i, \lambda_j)([\xi_{ij}])$. In this case, denote by $E_A^r(M_i, N_j)$ the image of $\mathrm{Ext}_A^r(M_i, N_j)$ under the map ψ. Then $\mathrm{Ext}_A^r(\oplus_{i=1}^s M_i, \oplus_{j=1}^t N_j) = \coprod_{i=1}^s \coprod_{j=1}^t E_A^r(M_i, N_j)$.

1.3 Morita invariance of quasi-Koszulity

Firstly, we have the following "expansion lemma":

Lemma 1 *Let M, N, L_i be A-modules with $1 \leq i \leq z$. Then, for $r, l \geq 0$, as the K-subspaces of $Ext_A^{r+l}(M, N)$,*

$$Ext_A^r(M, \oplus_{i=1}^z L_i) Ext_A^l(\oplus_{i=1}^z L_i, N) = \sum_{i=1}^z Ext_A^r(M, L_i) Ext_A^l(L_i, N).$$

Proof. For $[\xi] \in \mathrm{Ext}_A^r(M, \oplus_{i=1}^z L_i)$ and $[\zeta] \in \mathrm{Ext}_A^l(\oplus_{i=1}^z L_i, N)$, we have

$$\begin{array}{rl} & [\xi]\mathrm{Ext}_A^l(\lambda_i\rho_i, N)([\zeta]) \\ = & \mathrm{Ext}_A^r(M, \rho_i)([\xi])\mathrm{Ext}_A^l(\lambda_i, N)([\zeta]) \\ = & \mathrm{Ext}_A^r(M, \lambda_i\rho_i)([\xi])[\zeta]. \end{array}$$

Thus

$$\begin{array}{rcl} [\xi][\zeta] & = & [\xi]\mathrm{Ext}_A^l(\sum_{i=1}^z \lambda_i\rho_i, N)([\zeta]) \\ & = & \sum_{i=1}^z [\xi]\mathrm{Ext}_A^l(\lambda_i\rho_i, N)([\zeta]) \\ & = & \sum_{i=1}^z \mathrm{Ext}_A^r(M, \rho_i)([\xi])\mathrm{Ext}_A^l(\lambda_i, N)([\zeta]) \\ & \in & \sum_{i=1}^z \mathrm{Ext}_A^r(M, L_i)\mathrm{Ext}_A^l(L_i, N). \end{array}$$

Conversely, for $[\xi_i] \in \mathrm{Ext}_A^r(M, L_i)$ and $[\zeta_i] \in \mathrm{Ext}_A^l(L_i, N)$, we have

$$\begin{array}{rcl} [\xi_i][\zeta_i] & = & [\xi_i]\mathrm{Ext}_A^l(\rho_i\lambda_i, N)([\zeta_i]) \\ & = & \mathrm{Ext}_A^r(M, \lambda_i)([\xi_i])\mathrm{Ext}_A^l(\rho_i, N)([\zeta_i]) \\ & \in & \mathrm{Ext}_A^r(M, \oplus_{i=1}^z L_i)\mathrm{Ext}_A^l(\oplus_{i=1}^z L_i, N). \end{array}$$

$\square$

Secondly, we have the following "cancelation lemma":

Lemma 2 *Let M, N, L_1, L_2, L_3 be A-modules with $L_2 \cong L_3$. Then, for $r, l \geq 0$, as the K-subspaces of $Ext_A^{r+l}(M, N)$,*

$$Ext_A^r(M, \oplus_{i=1}^3 L_i)Ext_A^l(\oplus_{i=1}^3 L_i, N) = Ext_A^r(M, \oplus_{i=1}^2 L_i)Ext_A^l(\oplus_{i=1}^2 L_i, N).$$

Proof. Assume that $\sigma : L_2 \to L_3$ is an isomorphism of A-modules. For $[\xi_3] \in \mathrm{Ext}_A^r(M, L_3)$ and $[\zeta_3] \in \mathrm{Ext}_A^l(L_3, N)$, we have

$$\begin{array}{rcl} [\xi_3][\zeta_3] & = & [\xi_3]\mathrm{Ext}_A^l(\sigma\sigma^{-1}, N)([\zeta_3]) \\ & = & \mathrm{Ext}_A^r(M, \sigma^{-1})([\xi_3])\mathrm{Ext}_A^l(\sigma, N)([\zeta_3]) \\ & \in & \mathrm{Ext}_A^r(M, L_2)\mathrm{Ext}_A^l(L_2, N). \end{array}$$

By Lemma 1, we have

$$\begin{array}{rcl} \mathrm{Ext}_A^r(M, \oplus_{i=1}^3 L_i)\mathrm{Ext}_A^l(\oplus_{i=1}^3 L_i, N) & = & \sum_{i=1}^3 \mathrm{Ext}_A^r(M, L_i)\mathrm{Ext}_A^l(L_i, N) \\ & = & \sum_{i=1}^2 \mathrm{Ext}_A^r(M, L_i)\mathrm{Ext}_A^l(L_i, N) \\ & = & \mathrm{Ext}_A^r(M, \oplus_{i=1}^2 L_i)\mathrm{Ext}_A^l(\oplus_{i=1}^2 L_i, N). \end{array}$$

$\square$

For any K-algebra A, denote by Mod$A$ (resp. mod$A$) the category of (resp. finitely generated) left A-modules. We have the following "extension group isomorphism lemma":

Lemma 3 *Let F : Mod$A \to$ ModA' (resp. F : mod$A \to$ modA') be an equivalence functor. Then F induces K-vector space isomorphisms F_{MN} : $Ext_A^r(M, N) \to Ext_{A'}^r(F(M), F(N))$ for all M, N in Mod$A$ (resp. mod$A$) and $r \geq 0$.*

Proof. In case $r = 0$ the isomorphism is well-known (cf. [1, 21.2. Proposition]). By [1, 16.3. Proposition], $F(M_1 \oplus M_2)$ is a direct sum $F(M_1)$ and $F(M_2)$ with injections $F(\lambda_1)$ and $F(\lambda_2)$. Thus there are A'-modules maps $\phi_{M_1M_2} : F(M_1 \oplus M_2) \to F(M_1) \oplus F(M_2)$ and $\psi_{M_1M_2} : F(M_1) \oplus F(M_2) \to F(M_1 \oplus M_2)$ such that $\phi_{M_1M_2}\psi_{M_1M_2} = 1, \psi_{M_1M_2}\phi_{M_1M_2} = 1, \phi_{M_1M_2}F(\lambda_i) = \lambda_i$ and $F(\rho_i)\psi_{M_1M_2} = \rho_i$ for $i = 1, 2$. Hence $\phi_{MM}F(\Delta) = \Delta$ and $F(\nabla)\psi_{MM} = \nabla$.

Let $f_i : M_i \to N_i$ be A-module maps with $i = 1, 2$. Applying the formulae above, we can show that $\phi_{N_1N_2}F(diag\{f_1, f_2\})) = diag\{F(f_1), F(f_2)\}\phi_{M_1M_2}$, since $diag\{f_1, f_2\} = \lambda_1 f_1 \rho_1 + \lambda_2 f_2 \rho_2$.

The functor F induces a map $F_{MN} : \mathrm{Ext}_A^r(M, N) \to \mathrm{Ext}_{A'}^r(F(M), F(N))$, $[\xi] \to [F(\xi)]$. For $[\xi_1], [\xi_2] \in \mathrm{Ext}_A^r(M, N)$, using the preparations above we

can show routinely that $F_{MN}([\xi_1]+[\xi_2]) = F_{MN}([\xi_1]) + F_{MN}([\xi_2])$. So F_{MN} is a K-linear map.

Let G be an inverse equivalence of F. Then G also induces a K-linear map $G_{M'N'} : \mathrm{Ext}^r_{A'}(M', N') \to \mathrm{Ext}^r_A(G(M'), G(N')), [\zeta] \to [G(\zeta)]$.

The natural isomorphism η from the functor GF to the identity functor induces an isomorphism $\eta_{MN} : \mathrm{Ext}^r_A(GF(M), GF(N)) \to \mathrm{Ext}^r_A(M, N), [\xi] \mapsto [\eta(\xi)]$, where $\eta(\xi)$ is obtained from ξ by replacing the monomorphism (resp. epimorphism) in ξ with the composition of it and the isomorphism $\eta_N^{-1} : N \to GF(N)$ (resp. $\eta_M : GF(M) \to M$).

Since $\eta_{GF(M)GF(N)} G_{F(M)F(N)} F_{MN} = 1$, $G_{F(M)F(N)} F_{MN}$ is a bijection. Similarly, $F_{GF(M)GF(N)} G_{F(M)F(N)}$ is a bijection. Hence $G_{F(M)F(N)}$ is a bijection. It follows that F_{MN} is a bijection. $\square$

Theorem 1 *Let $F : ModA \to ModA'$ or $F : modA \to modA'$ be an equivalence functor. If A is a quasi-Koszul algebra then so is A'. If M is a quasi-Koszul A-module then $F(M)$ is a quasi-Koszul A'-module.*

Proof. By [1, 21.8. Proposition and 27.8. Corollary], A' is also noetherian and semiperfect. Denote by $S_1, ..., S_s$ a complete set of the representatives of simple A-modules. Then $S_1' := F(S_1), ..., S_s' := F(S_s)$ is a complete set of the representatives of simple A'-modules (cf. [1, §21 and §27]). Assume that $A/J \cong \oplus_{i=1}^s S_i^{u_i} = \oplus_{i=1}^s \oplus_{k=1}^{u_i} S_{ik}$ and $A'/J' \cong \oplus_{j=1}^s S_j'^{u_j'} = \oplus_{j=1}^s \oplus_{l=1}^{u_j'} S'_{jl}$ where J' is the Jacobson radical of A', $S_{ik} = S_i$ and $S'_{jl} = S'_j$ for all i, j, k, l.

The algebra A is quasi-Koszul, by definition, $\mathrm{Ext}^*_A(A/J, A/J)$ is generated in degree 0 and 1. So is $\mathrm{Ext}^*_A(\oplus_{i=1}^s S_i^{u_i}, \oplus_{i=1}^s S_i^{u_i}) = \mathrm{Ext}^*_A(\oplus_{i=1}^s \oplus_{j=1}^{u_i} S_{ij}, \oplus_{i=1}^s \oplus_{j=1}^{u_i} S_{ij})$. Thus

$$\begin{array}{rl} & \mathrm{Ext}^{r+1}_A(\oplus_{i=1}^s \oplus_{j=1}^{u_i} S_{ij}, \oplus_{i=1}^s \oplus_{j=1}^{u_i} S_{ij}) \\ = & \mathrm{Ext}^r_A(\oplus_{i=1}^s \oplus_{j=1}^{u_i} S_{ij}, \oplus_{i=1}^s \oplus_{j=1}^{u_i} S_{ij}) \mathrm{Ext}^1_A(\oplus_{i=1}^s \oplus_{j=1}^{u_i} S_{ij}, \oplus_{i=1}^s \oplus_{j=1}^{u_i} S_{ij}) \end{array}$$

for all $r \geq 1$. By Section 1.2, we have

$$\begin{array}{rl} & \coprod_{i=1}^s \coprod_{j=1}^{u_i} \coprod_{k=1}^s \coprod_{l=1}^{u_k} E^{r+1}_A(S_{ij}, S_{kl}) \\ = & \sum_{i=1}^s \sum_{j=1}^{u_i} \sum_{k=1}^s \sum_{l=1}^{u_k} E^r_A(S_{ij}, \oplus_{p=1}^s \oplus_{q=1}^{u_p} S_{pq}) E^1_A(\oplus_{p=1}^s \oplus_{q=1}^{u_p} S_{pq}, S_{kl}) \end{array}$$

for all $r \geq 1$. Note that $E^r_A(S_{ij}, \oplus_{p=1}^s \oplus_{q=1}^{u_p} S_{pq}) E^1_A(\oplus_{p=1}^s \oplus_{q=1}^{u_p} S_{pq}, S_{kl}) \subseteq E^{r+1}_A(S_{ij}, S_{kl})$. Thus

$$E^{r+1}_A(S_{ij}, S_{kl}) = E^r_A(S_{ij}, \oplus_{p=1}^s \oplus_{q=1}^{u_p} S_{pq}) E^1_A(\oplus_{p=1}^s \oplus_{q=1}^{u_p} S_{pq}, S_{kl})$$

for all i, j, k, l, r. It follows

$$\mathrm{Ext}^{r+1}_A(S_{ij}, S_{kl}) = \mathrm{Ext}^r_A(S_{ij}, \oplus_{p=1}^s \oplus_{q=1}^{u_p} S_{pq}) \mathrm{Ext}^1_A(\oplus_{p=1}^s \oplus_{q=1}^{u_p} S_{pq}, S_{kl})$$

for all i, j, k, l, r. By Lemma 2, we obtain

$$\mathrm{Ext}_A^{r+1}(S_{ij}, S_{kl}) = \mathrm{Ext}_A^r(S_{ij}, \oplus_{p=1}^s S_p)\mathrm{Ext}_A^1(\oplus_{p=1}^s S_p, S_{kl})$$

for all i, j, k, l, r. Applying Lemma 3 and Lemma 2, we have

$$\begin{array}{rl} & \mathrm{Ext}_{A'}^{r+1}(S'_{ij}, S'_{kl}) \\ = & F_{S_{ij}S_{kl}}(\mathrm{Ext}_A^{r+1}(S_{ij}, S_{kl})) \\ = & F_{S_{ij}S_{kl}}(\mathrm{Ext}_A^r(S_{ij}, \oplus_{p=1}^s S_p)\mathrm{Ext}_A^1(\oplus_{p=1}^s S_p, S_{kl})) \\ = & \mathrm{Ext}_{A'}^r(S'_{ij}, F(\oplus_{p=1}^s S_p))\mathrm{Ext}_{A'}^1(F(\oplus_{p=1}^s S_p), S'_{kl})) \\ = & \mathrm{Ext}_{A'}^r(S'_{ij}, \oplus_{p=1}^s S'_p)\mathrm{Ext}_{A'}^1(\oplus_{p=1}^s S'_p, S'_{kl}) \\ = & \mathrm{Ext}_{A'}^r(S'_{ij}, \oplus_{p=1}^s \oplus_{q=1}^{u'_p} S'_{pq})\mathrm{Ext}_{A'}^1(\oplus_{p=1}^s \oplus_{q=1}^{u'_p} S'_{pq}, S'_{kl}) \end{array}$$

for all i, j, k, l, r. It follows

$$E_{A'}^{r+1}(S'_{ij}, S'_{kl}) = E_{A'}^r(S'_{ij}, \oplus_{p=1}^s \oplus_{q=1}^{u'_p} S'_{pq})E_{A'}^1(\oplus_{p=1}^s \oplus_{q=1}^{u'_p} S'_{pq}, S'_{kl})$$

for all i, j, k, l, r. Thus

$$\begin{array}{rl} & \coprod_{i=1}^s \coprod_{j=1}^{u'_i} \coprod_{k=1}^s \coprod_{l=1}^{u'_k} E_{A'}^{r+1}(S'_{ij}, S'_{kl}) \\ = & \sum_{i=1}^s \sum_{j=1}^{u'_i} \sum_{k=1}^s \sum_{l=1}^{u'_k} E_{A'}^r(S'_{ij}, \oplus_{p=1}^s \oplus_{q=1}^{u'_p} S'_{pq})E_{A'}^1(\oplus_{p=1}^s \oplus_{q=1}^{u'_p} S'_{pq}, S'_{kl}) \end{array}$$

for all $r \geq 1$. It follows from Section 1.2 that

$$\begin{array}{rl} & \mathrm{Ext}_{A'}^{r+1}(\oplus_{i=1}^s \oplus_{j=1}^{u'_i} S'_{ij}, \oplus_{i=1}^s \oplus_{j=1}^{u'_i} S'_{ij}) \\ = & \mathrm{Ext}_{A'}^r(\oplus_{i=1}^s \oplus_{j=1}^{u'_i} S'_{ij}, \oplus_{i=1}^s \oplus_{j=1}^{u'_i} S'_{ij})\mathrm{Ext}_{A'}^1(\oplus_{i=1}^s \oplus_{j=1}^{u'_i} S'_{ij}, \oplus_{i=1}^s \oplus_{j=1}^{u'_i} S'_{ij}) \end{array}$$

for all $r \geq 1$. So $\mathrm{Ext}_{A'}^*(\oplus_{i=1}^s S_i'^{u'_i}, \oplus_{i=1}^s S_i'^{u'_i}) = \mathrm{Ext}_{A'}^*(\oplus_{i=1}^s \oplus_{j=1}^{u'_i} S'_{ij}, \oplus_{i=1}^s \oplus_{j=1}^{u'_i} S'_{ij})$ is generated in degree 0 and 1. Thus $\mathrm{Ext}_{A'}^*(A'/J', A'/J')$ is generated in degree 0 and 1, i.e., A' is quasi-Koszul.

Now suppose M is a quasi-Koszul A-module, we show that $F(M)$ is a quasi-Koszul A'-module. By definition,

$$\mathrm{Ext}_A^{r+1}(M, A/J) = \mathrm{Ext}_A^r(M, A/J)\mathrm{Ext}_A^1(A/J, A/J)$$

for all $r \geq 0$. It follows from Lemma 3 that

$$\mathrm{Ext}_{A'}^{r+1}(F(M), F(A/J)) = \mathrm{Ext}_{A'}^r(F(M), F(A/J))\mathrm{Ext}_{A'}^1(F(A/J), F(A/J))$$

for all $r \geq 0$. Therefore

$$\begin{array}{rl} & \mathrm{Ext}_{A'}^{r+1}(F(M), \oplus_{i=1}^s \oplus_{j=1}^{u_i} S'_{ij}) \\ = & \mathrm{Ext}_{A'}^r(F(M), \oplus_{i=1}^s \oplus_{j=1}^{u_i} S'_{ij})\mathrm{Ext}_{A'}^1(\oplus_{i=1}^s \oplus_{j=1}^{u_i} S'_{ij}, \oplus_{i=1}^s \oplus_{j=1}^{u_i} S'_{ij}) \end{array}$$

for all $r \geq 0$. By Section 1.2, we have

$$\begin{aligned} & \coprod_{i=1}^{s} \coprod_{j=1}^{u_i} E_{A'}^{r+1}(F(M), S'_{ij}) \\ = & \sum_{i=1}^{s} \sum_{j=1}^{u_i} \mathrm{Ext}_{A'}^{r}(F(M), \oplus_{p=1}^{s} \oplus_{q=1}^{u_p} S'_{pq}) E_{A'}^{1}(\oplus_{p=1}^{s} \oplus_{q=1}^{u_p} S'_{pq}, S'_{ij}) \end{aligned}$$

for all $r \geq 0$. Note that $\mathrm{Ext}_{A'}^{r}(F(M), \oplus_{p=1}^{s} \oplus_{q=1}^{u_p} S'_{pq}) E_{A'}^{1}(\oplus_{p=1}^{s} \oplus_{q=1}^{u_p} S'_{pq}, S'_{ij}) \subseteq E_{A'}^{r+1}(F(M), S'_{ij})$. Thus

$$E_{A'}^{r+1}(F(M), S'_{ij}) = \mathrm{Ext}_{A'}^{r}(F(M), \oplus_{p=1}^{s} \oplus_{q=1}^{u_p} S'_{pq}) E_{A'}^{1}(\oplus_{p=1}^{s} \oplus_{q=1}^{u_p} S'_{pq}, S'_{ij})$$

for all i, j, r. It follows

$$\mathrm{Ext}_{A'}^{r+1}(F(M), S'_{ij}) = \mathrm{Ext}_{A'}^{r}(F(M), \oplus_{p=1}^{s} \oplus_{q=1}^{u_p} S'_{pq}) \mathrm{Ext}_{A'}^{1}(\oplus_{p=1}^{s} \oplus_{q=1}^{u_p} S'_{pq}, S'_{ij})$$

for all i, j, r. By Lemma 2, we have

$$\begin{aligned} & \mathrm{Ext}_{A'}^{r+1}(F(M), S'_{ij}) \\ = & \mathrm{Ext}_{A'}^{r}(F(M), \oplus_{p=1}^{s} S'_{p}) \mathrm{Ext}_{A'}^{1}(\oplus_{p=1}^{s} S'_{p}, S'_{i}) \\ = & \mathrm{Ext}_{A'}^{r}(F(M), \oplus_{p=1}^{s} \oplus_{q=1}^{u'_p} S'_{pq}) \mathrm{Ext}_{A'}^{1}(\oplus_{p=1}^{s} \oplus_{q=1}^{u'_p} S'_{pq}, S'_{ij}) \end{aligned}$$

for all i, j, r. It follows

$$E_{A'}^{r+1}(F(M), S'_{ij}) = \mathrm{Ext}_{A'}^{r}(F(M), \oplus_{p=1}^{s} \oplus_{q=1}^{u'_p} S'_{pq}) E_{A'}^{1}(\oplus_{p=1}^{s} \oplus_{q=1}^{u'_p} S'_{pq}, S'_{ij})$$

for all i, j, r. Thus

$$\begin{aligned} & \coprod_{i=1}^{s} \coprod_{j=1}^{u'_i} E_{A'}^{r+1}(F(M), S'_{ij}) \\ = & \sum_{i=1}^{s} \sum_{j=1}^{u'_i} \mathrm{Ext}_{A'}^{r}(F(M), \oplus_{p=1}^{s} \oplus_{q=1}^{u'_p} S'_{pq}) E_{A'}^{1}(\oplus_{p=1}^{s} \oplus_{q=1}^{u'_p} S'_{pq}, S'_{ij}) \end{aligned}$$

for all $r \geq 0$. By Section 1.2, we obtain

$$\begin{aligned} & \mathrm{Ext}_{A'}^{r+1}(F(M), \oplus_{i=1}^{s} \oplus_{j=1}^{u'_i} S'_{ij}) \\ = & \mathrm{Ext}_{A'}^{r}(F(M), \oplus_{i=1}^{s} \oplus_{j=1}^{u'_i} S'_{ij}) \mathrm{Ext}_{A'}^{1}(\oplus_{i=1}^{s} \oplus_{j=1}^{u'_i} S'_{ij}, \oplus_{i=1}^{s} \oplus_{j=1}^{u'_i} S'_{ij}) \end{aligned}$$

for all $r \geq 0$. Thus

$$\mathrm{Ext}_{A'}^{r+1}(F(M), A'/J') = \mathrm{Ext}_{A'}^{r}(F(M), A'/J') \mathrm{Ext}_{A'}^{1}(A'/J', A'/J')$$

for all $r \geq 0$, i.e., $F(M)$ is a quasi-Koszul A'-module. $\square$

Remark 1 *Though the quasi-Koszulity of algebras is invariant under Morita equivalence, it is not so under derived equivalence even tilting equivalence in general. Indeed, all path algebras are Koszul thus quasi-Koszul. If all their tilted algebras were quasi-Koszul then, as length graded algebras, all monomial tilted algebras would be Koszul, thus quadratic [17, Corollary 7.3]. It is a contradiction [31, Appendix]. Nevertheless, since the global dimension of tilted algebras are less equal to 2 [19, Theorem (5.2)], all quadratic tilted algebras are Koszul [17, Theorem 7.2].*

Corollary 1 *A is quasi-Koszul if and only if so is $M_n(A)$.*

2 Skew group algebras and quasi-Koszulity

In this section, we show that, a finite-dimensional K-algebra A with an action of G is quasi-Koszul if and only if so is the skew group algebra $A*G$, where G is a finite group satisfying $\mathrm{char}K \nmid |G|$. Moreover, we prove that the Yoneda algebra $\mathrm{Ext}^*_{A*G}((A*G)/J(A*G),(A*G)/J(A*G))$ of the skew group algebra $A*G$ is isomorphic to the skew group algebra $\mathrm{Ext}^*_A(A/J,A/J)*G$ of the Yoneda algebra $\mathrm{Ext}^*_A(A/J,A/J)$. The results of this section are essentially due to Martínez-Villa (cf. [25]).

2.1 Skew group algebras

Let A be a K-algebra and G a group of K-algebra automorphisms of A. Then the *skew group algebra* $A*G$ is defined as follows: As a K-vector space $A*G = A\otimes_K KG$. For $a\in A$ and $g\in G$, write ag instead of $a\otimes g$, and define the multiplication by $ga := g(a)g$.

Let A be a K-algebra and G a group. Then an *action* of G on A is a map $G\times A\to A, (g,a)\mapsto ga$ satisfying $(g_1g_2)a = g_1(g_2a), 1a = a, g(a_1+a_2) = ga_1+ga_2, g(a_1a_2) = (ga_1)(ga_2), g(ka) = k(ga)$, for all $g, g_1, g_2 \in G, a, a_1, a_2 \in A, k\in K$.

For a K-algebra A, giving a group G of K-automorphisms of A is equivalent to giving an action of a group G on A, or giving a group homomorphism $G\to \mathrm{Aut}_k(A)$ where $\mathrm{Aut}_k(A)$ denotes the group of K-algebra automorphisms of A, or giving a KG-module algebra structure (cf. [11, Proposition 1.2]).

2.2 Quasi-Koszulity of skew group algebras

Let G be a finite group and M a KG-module. Denote by M^G the KG-module $\{m\in M | gm = m \text{ for all } g\in G\}$. If $\mathrm{char}K\nmid |G|$ then the fixed point functor $(-)^G : \mathrm{Mod}KG\to \mathrm{Mod}KG$ is exact (cf. [25, Lemma 3]).

Lemma 4 *(cf. [25, Lemma 4]) Let A be a K-algebra, G a finite group acting on A, and M, N, L three $A*G$-modules. Then*

(1) $Hom_A(M,N)$ is a KG-module defined by $(g\phi)(m) = g\phi(g^{-1}m)$ for $g\in G, \phi\in Hom_A(M,N), m\in M$.

(2) $Hom_A(M,N)^G = Hom_{A*G}(M,N)$.

(3) $Ext^i_A(M,N)$ is a KG-module satisfying $g([\xi][\zeta]) = (g[\xi])(g[\zeta])$ for $g\in G$, $[\xi]\in Ext^j_A(L,M)$ and $[\zeta]\in Ext^i_A(M,N)$.

Proof. (1), (2): Easy to check.

(3): To each $g\in G$, we associate a functor $(-)^g : \mathrm{Mod}A\to \mathrm{Mod}A$. For each X in $\mathrm{Mod}A$, X^g is defined as follows: As a K-vector space $X^g = X$. For

$a \in A$ and $x \in X^g$, $a \cdot x := (ga)x$. For X, Y in Mod$A$ and $\phi \in \mathrm{Hom}_A(X,Y)$, $\phi^g := \phi$. Obviously, the functor $(-)^g$ is not only an exact functor but also an automorphism of Mod$A$ with the inverse $()^{g^{-1}}$.

Since M is an $A*G$-module, we have an A-module isomorphism $\Psi^g_M : M \to M^g, m \mapsto gm$ with the inverse $\Psi^{g^{-1}}_{M^g} : M^g \to M$ (cf. [22, Proposition 2.5]). Note that for $\psi \in \mathrm{Hom}_A(M,N)$ we have $\Psi^g_{N^{g^{-1}}}\psi^{g^{-1}}\Psi^{g^{-1}}_M = g\psi$.

For $g \in G$ and $[\xi] \in \mathrm{Ext}^i_A(M,N)$, the exact functor $(-)^{g^{-1}}$ provides an i-extension $\xi^{g^{-1}}$ of $M^{g^{-1}}$ by $N^{g^{-1}}$. Since M and N are $A*G$-modules, replacing the epimorphism (resp. monomorphism) in $\xi^{g^{-1}}$ with the composition of it and the A-module isomorphism $\Psi^g_{M^{g^{-1}}}$ (resp. $\Psi^{g^{-1}}_N$), we obtain an i-extension $g\xi$ of M by N. One can shows that $\mathrm{Ext}^i_A(M,N)$ is a KG-module defined by $g[\xi] := [g\xi]$ and satisfying $g([\xi][\zeta]) = (g[\xi])(g[\zeta])$ for $g \in G$, $[\xi] \in \mathrm{Ext}^j_A(L,M)$ and $[\zeta] \in \mathrm{Ext}^i_A(M,N)$. $\square$

Lemma 5 *(cf. [25, Corollary 5]) Let A be a K-algebra, G a finite group acting on A, and $\varsigma : 0 \to L \xrightarrow{\phi} M \xrightarrow{\psi} N \to 0$ an exact sequence of $A*G$-modules. Then for any $A*G$-module X two long exact sequences*

$$\begin{array}{rl} 0 \to & Hom_A(X,L) \to Hom_A(X,M) \to Hom_A(X,N) \\ \to & Ext^1_A(X,L) \to Ext^1_A(X,M) \to Ext^1_A(X,N) \to \cdots \end{array}$$

$$\begin{array}{rl} 0 \to & Hom_A(N,X) \to Hom_A(M,X) \to Hom_A(L,X) \\ \to & Ext^1_A(N,X) \to Ext^1_A(M,X) \to Ext^1_A(L,X) \to \cdots \end{array}$$

are exact sequences of KG-modules and KG-module maps.

Proof. By Lemma 4 (1) and (3), we know that all terms in the long exact sequences are KG-modules.

Since ϕ is an $A*G$-module map, by Lemma 4 (2), we have $g\phi = \phi$ for all $g \in G$. For any $[\xi] \in \mathrm{Ext}^i_A(X,L)$, using $\Psi^g_{M^{g^{-1}}}\phi^{g^{-1}}\Psi^{g^{-1}}_L = g\phi$, we can show that $g\mathrm{Ext}^i_A(X,\phi)([\xi]) = \mathrm{Ext}^i_A(X,g\phi)([g\xi]) = \mathrm{Ext}^i_A(X,\phi)(g[\xi])$. It follows that $\mathrm{Ext}^i_A(X,\phi)$ is a KG-module map. Similar for $\mathrm{Ext}^i_A(X,\psi)$.

Let $\delta : \mathrm{Ext}^i_A(X,N) \to \mathrm{Ext}^{i+1}_A(X,L)$ be the connecting map and $[\zeta] \in \mathrm{Ext}^i_A(X,N)$. Since ς is an exact sequence of $A*G$-modules, we have $g[\varsigma] = [\varsigma]$. By Lemma 4 (3), we have $\delta(g[\zeta]) = (g[\zeta])[\varsigma] = (g[\zeta])(g[\varsigma]) = g([\zeta\varsigma]) = g\delta([\zeta])$. It follows that δ is a KG-module map.

Hence the first exact sequence is an exact sequence of KG-modules and KG-module maps. Similar for the second exact sequence. $\square$

Lemma 6 *(cf. [25, Lemma 6]) Let A be a K-algebra, G a finite group acting on A with char$K \nmid |G|$ and P a finitely generated $A*G$-module. Then P is projective if and only if P is projective as an A-module.*

Proof. If P is projective as an $A*G$-module then P is a direct summand of a free $A*G$-module $(A*G)^i$. So P is a direct summand of a free A-module $(A*G)^i \cong A^{|G|i}$. Hence P is a projective A-module.

Conversely, if P is a projective A-module then by Lemma 5 the functor $\mathrm{Hom}_A(P,-) : \mathrm{Mod}A*G \to \mathrm{Mod}KG$ is exact. Since the functor $(-)^G : \mathrm{Mod}KG \to \mathrm{Mod}KG$ is exact, the functor $\mathrm{Hom}_A(P,-)^G : \mathrm{Mod}A*G \to \mathrm{Mod}KG$ is also exact. By Lemma 4 (2) the functor $\mathrm{Hom}_{A*G}(P,-) = \mathrm{Hom}_A(P,-)^G : \mathrm{Mod}A*G \to \mathrm{Mod}KG$ is exact. Thus P is also projective as an $A*G$-module. $\square$

Let M be an $A*G$-module and W a KG-module. Then $M \otimes_K W$ is an $A*G$-module defined by $(ag)(m \otimes w) = agm \otimes gw$ for $a \in A, g \in G, m \in M, w \in W$.

Lemma 7 *(cf. [25, Lemma 8]) Let A be a K-algebra, G a finite group acting on A, P a finitely generated projective $A*G$-module, N an $A*G$-module and W a KG-module. Then we have a K-vector space isomorphism $\theta_P : Hom_A(P,N) \otimes_K W \to Hom_{A*G}(P \otimes_K KG, N \otimes_K W)$ defined by $\theta_P(\phi \otimes w)(p \otimes g) := (g\phi)(p) \otimes gw$.*

Proof. Consider the natural isomorphisms $\sigma : \mathrm{Hom}_A(A,N) \otimes_K W \to N \otimes_K W, \phi \otimes w \mapsto \phi(1) \otimes w$ and $\tau : \mathrm{Hom}_{A*G}(A \otimes_K KG, N \otimes_K W) \to N \otimes_K W, \psi \mapsto \psi(1 \otimes 1)$. We have $\theta_A = \tau^{-1}\sigma$. Thus Lemma 7 holds for $P = A$. Next we can prove routinely that Lemma 7 holds for all A^n with $n \geq 1$ and all finitely generated projective $A*G$-module P. $\square$

Lemma 8 *(cf. [25, Proposition 9]) Let A be a K-algebra, G a finite group acting on A with $charK \nmid |G|$, M an $A*G$-module admitting a finitely generated $A*G$-module projective resolution $\mathbb{P} = (P_i, \psi_i)_{i\geq 0}$, N an $A*G$-module and W a KG-module. Then we have K-vector space isomorphisms $\bar{\theta}_i : Ext^i_A(M,N)\otimes_K W \to Ext^i_{A*G}(M\otimes_K KG, N\otimes_K W)$ defined by $\bar{\theta}_i(\bar{\phi}\otimes w) := \overline{\theta_{P_i}(\phi \otimes w)}$, for all $i \geq 0$.*

Remark 2 *Here we view the elements in $Ext^i_A(M,N)$ as the residue classes in $Hom_A(P_i,N)/ImHom_A(\psi_i,N)$ or $Hom_A(\Omega^i(M),N)/ImHom_A(\iota_i,N)$ where $\Omega^i(M)$ denotes the i-th syzygy of M and $\iota_i : \Omega^i(M) \hookrightarrow P_{i-1}$ is the natural embedding (cf. [21, Chapter III, Theorem 6.4]), because in this situation this viewpoint is more intuitive.*

Proof of Lemma 8. By Lemma 6, $\mathbb{P}$ is also a finitely generated A-module projective resolution of M. Thus $\mathbb{P} \otimes_K KG$ is an exact sequence of $A*G$-modules where the $A*G$-module structure of $P_i \otimes_K KG$ is given

by $(ag)(p_i \otimes h) = agp_i \otimes gh$. By Lemma 6 again, $\mathbb{P} \otimes_K KG = (P_i \otimes_K KG, \psi_i \otimes 1)_{i \geq 0}$ is a finitely generated $A * G$-module projective resolution of $M \otimes_K KG$. It follows from Lemma 7 that $(\theta_{P_i})_{i \geq 0}$ is a chain isomorphism from the complex $\mathrm{Hom}_A(\mathbb{P}, N) \otimes_K W$ to the complex $\mathrm{Hom}_A(\mathbb{P} \otimes_K KG, N \otimes_K W)$. This chain isomorphism induces isomorphisms $\bar{\theta}_i : \mathrm{Ext}^i_A(M, N) \otimes_K W \to \mathrm{Ext}^i_{A*G}(M \otimes_K KG, N \otimes_K W)$ between the homologies of these complexes defined by $\bar{\theta}_i(\bar{\phi} \otimes w) := \overline{\theta_{P_i}(\phi \otimes w)}$ for all $i \geq 1$. The case of $i = 0$ follows from Five Lemma. $\square$

Theorem 2 *(cf. [25, Theorem 10]) Let A be a finite-dimensional K-algebra, G a finite group acting on A with char$K \nmid |G|$ and M a finitely generated $A*G$-module. Then $M \otimes_K KG$ is a quasi-Koszul $A*G$-module if and only if M is a quasi-Koszul A-module. In particular, the algebra $A*G$ is quasi-Koszul if and only if so is A. Moreover, $Ext^*_{A*G}((A*G)/J(A*G), (A*G)/J(A*G)) \cong Ext^*_A(A/J, A/J) * G$ as positively graded K-algebras.*

Proof. Let $\mathbb{P} = (P_k, \psi_k)_{k \geq 0}$ be a finitely generated $A * G$-module projective resolution of M. First of all, we show that the following diagram is commutative for all $i, j \geq 0$:

$$\begin{array}{ccc}
(\mathrm{Ext}^i_A(M, A/J) \otimes_K KG) \times (\mathrm{Ext}^j_A(A/J, A/J) \otimes_K KG) & \xrightarrow{\mu} & \mathrm{Ext}^{i+j}_A(M, A/J) \otimes_K KG \\
\downarrow (\bar{\theta}_i, \bar{\theta}_j) & & \downarrow \bar{\theta}_{i+j} \\
\mathrm{Ext}^i_{A*G}(M \otimes_K KG, (A/J) * G) \times \mathrm{Ext}^j_{A*G}((A/J) * G, (A/J) * G) & \xrightarrow{\nu} & \mathrm{Ext}^{i+j}_{A*G}(M \otimes_K KG, (A/J) * G)
\end{array}$$

where ν is Yoneda product and μ is defined by $\mu(\bar{\alpha} \otimes g, \bar{\beta} \otimes h) := \bar{\alpha}(g\bar{\beta}) \otimes gh$.

Let $\mathbb{Q} = (Q_k, \phi_k)_{k \geq 0}$ be a finitely generated $A * G$-module projective resolution of A/J. Let $\bar{\alpha} \in \mathrm{Ext}^i_A(M, A/J)$ and $\bar{\beta} \in \mathrm{Ext}^j_A(A/J, A/J)$ with $\alpha \in \mathrm{Hom}_A(\Omega^i(M), A/J)$ or $\mathrm{Hom}_A(P_i, A/J)$ and $\beta \in \mathrm{Hom}_A(\Omega^j(A/J), A/J)$ or $\mathrm{Hom}_A(Q_j, A/J)$ (cf. Remark 2). Then $\bar{\alpha}\bar{\beta} = \overline{\beta\Omega^j(\alpha)}$.

Since $\mathbb{P} = (P_k, \psi_k)_{k \geq 0}$ is an $A * G$-module projective resolution of M, $\mathbb{P}(i) := (P_{k+i}, \psi_{k+i})_{k \geq 0}$ is an $A * G$-module projective resolution of $\Omega^i(M)$ for all $i \geq 0$. Furthermore, $\mathbb{P}(i) \otimes_K KG$ is an $A * G$-module projective resolution of $\Omega^i(M) \otimes_K KG$ for all $i \geq 0$. Suppose the map $\alpha : \Omega^i(M) \to A/J$ induces a chain map $(\alpha_j)_{j \geq 0} : \mathbb{P}(i) \to \mathbb{Q}$. Then $(\theta_{P_{j+i}}(\alpha_j \otimes g))_{j \geq 0} : \mathbb{P}(i) \otimes_K KG \to \mathbb{Q} \otimes_K KG$ is exactly a chain map induced by the map $\theta_{P_i}(\alpha \otimes g) : P_i \otimes_K KG \to (A/J) \otimes_K KG$. It follows that $\Omega^j(\theta_{P_i}(\alpha \otimes g)) = \theta_{P_{j+i}}(\Omega^j(\alpha) \otimes g)$. Thus for $x \otimes l \in P_{j+i} \otimes_K KG$

$$\begin{aligned}
& \theta_{Q_j}(\beta \otimes h)\Omega^j(\theta_{P_i}(\alpha \otimes g))(x \otimes l) \\
= \; & \theta_{Q_j}(\beta \otimes h)\theta_{P_{j+i}}(\Omega^j(\alpha) \otimes g))(x \otimes l) \\
= \; & \theta_{Q_j}(\beta \otimes h)((l\Omega^j(\alpha))(x) \otimes lg) \\
= \; & (lg\beta)((l\Omega^j(\alpha))(x)) \otimes lgh \\
= \; & lg\beta((lg)^{-1}l(\Omega^j(\alpha)(l^{-1}x))) \otimes lgh \\
= \; & l(g\beta(g^{-1}(\Omega^j(\alpha)(l^{-1}x)))) \otimes lgh \\
= \; & (l((g\beta)\Omega^j(\alpha)))(x) \otimes lgh \\
= \; & \theta_{P_{j+i}}((g\beta)\Omega^j(\alpha) \otimes gh)(x \otimes l),
\end{aligned}$$

i.e., $\theta_{Q_j}(\beta\otimes h)\Omega^j(\theta_{P_i}(\alpha\otimes g)) = \theta_{P_{j+i}}((g\beta)\Omega^j(\alpha)\otimes gh)$.

Therefore $\bar{\theta}_{i+j}((\bar{\alpha}\otimes g)(\bar{\beta}\otimes h)) = \bar{\theta}_{i+j}(\bar{\alpha}(g\bar{\beta})\otimes gh) = \bar{\theta}_{i+j}(\overline{(g\beta)\Omega^j(\alpha)}\otimes gh) = \overline{\theta_{P_{j+i}}((g\beta)\Omega^j(\alpha)\otimes gh)} = \overline{\theta_{Q_j}(\beta\otimes h)\Omega^j(\theta_{P_i}(\alpha\otimes g))} = \overline{\theta_{P_i}(\alpha\otimes g)}\,\overline{\theta_{Q_j}(\beta\otimes h)} = \bar{\theta}_i(\bar{\alpha}\otimes g)\bar{\theta}_j(\bar{\beta}\otimes h)$. So the diagram above is commutative.

If $\mathrm{char}K \nmid |G|$ then, by [30, Theorem 1.1], $J * G$ is the Jacobson radical of $A * G$. Moreover, $(A * G)/(J * G) \cong (A/J) * G$. Thus $M \otimes_K KG$ is a quasi-Koszul $A * G$-module if and only if

$$\begin{aligned} & \mathrm{Ext}^{i+j}_{A*G}(M\otimes_K KG, (A/J)*G) \\ = \; & \mathrm{Ext}^{i}_{A*G}(M\otimes_K KG, (A/J)*G)\mathrm{Ext}^{j}_{A*G}((A/J)*G, (A/J)*G) \end{aligned}$$

for all $i, j \geq 0$, if and only if

$$\begin{aligned} & \mathrm{Ext}^{i+j}_{A}(M, A/J)\otimes_K KG \\ = \; & (\mathrm{Ext}^{i}_{A}(M, A/J)\otimes_K KG)(\mathrm{Ext}^{j}_{A}(A/J, A/J)\otimes_K KG) \\ = \; & (\mathrm{Ext}^{i}_{A}(M, A/J)\mathrm{Ext}^{j}_{A}(A/J, A/J))\otimes_K KG \end{aligned}$$

for all $i, j \geq 0$, if and only if

$$\mathrm{Ext}^{i+j}_{A}(M, A/J) = \mathrm{Ext}^{i}_{A}(M, A/J)\mathrm{Ext}^{j}_{A}(A/J, A/J)$$

for all $i, j \geq 0$, if and only if M is a quasi-Koszul A-module. In particular, taking $M = A/J$ we have the algebra $A * G$ is quasi-Koszul if and only if so is A. Moreover, according to the commutative diagram above, the map $(\bar{\theta}_i)_{i\geq 0}$ is a positively graded K-algebra isomorphism $\mathrm{Ext}^*_A(A/J, A/J) * G \cong \mathrm{Ext}^*_{A*G}((A*G)/J(A*G), (A*G)/J(A*G))$. □

3 Smash product and quasi-Koszulity

In this section we show in two ways that a finite-dimensional G-graded K-algebra A is quasi-Koszul if and only if so is the smash product $A\#G^*$, where G is a finite group satisfying $\mathrm{char}K \nmid |G|$. Moreover, we prove that the Yoneda algebra $\mathrm{Ext}^*_{A\#G^*}((A\#G^*)/J(A\#G^*), (A\#G^*)/J(A\#G^*))$ of the smash product $A\#G^*$ is isomorphic to the smash product $\mathrm{Ext}^*_A(A/J, A/J)\#G^*$ of the Yoneda algebra $\mathrm{Ext}^*_A(A/J, A/J)$ as positively graded K-algebras.

3.1 Smash product

Let G be a finite group and $A = \coprod_{g\in G} A_g$ be a G-graded K-algebra. A *graded A-module* M is an A-module, together with a direct sum decomposition $M = \coprod_{g\in G} M_g$ of K-vector spaces such that $A_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Denote

by GrA the category whose objects are all graded A-modules and whose morphisms $\phi : M \to N$ are morphisms in Mod$A$ such that $\phi(M_g) \subset N_g$ for all $g \in G$. The category GrA is not only an abelian category but also a Grothendieck category (cf. [28, Section 2.2]). Since it is closed under kernel and has enough projective objects, each object in GrA has a graded projective resolution.

Let KG^* be the dual algebra of KG and $\{p_g|g \in G\}$ its dual basis, i.e., for $g \in G$ and $x = \sum_{h\in G} a_h h \in KG$ one has $p_g(x) = a_g \in K$ and $p_g p_h = \delta_{gh} p_h$ where δ_{gh} is the Kronecker delta. The *smash product*, denoted by $A\#G^*$ (cf. [11, Section 1] and [5, Section 2]), is the K-vector space $A \otimes_K KG^*$ with the multiplication given by $(a\#p_g)(b\#p_h) := ab_{gh^{-1}}\#p_h$ where $a\#p_g$ denotes the element $a \otimes p_g$.

For graded A-modules M and N, we denote by $\mathrm{HOM}_A(M, N)$ the set $\oplus_{g\in G}\mathrm{Hom}_A(M, N)_g$, where $\mathrm{Hom}_A(M, N)_g = \{f \in \mathrm{Hom}_A(M, N)|f(M_h) \subset N_{hg}$ for all $h \in G\}$. We write $\mathrm{EXT}^i_A(M, N)$ for the derived functor of $\mathrm{HOM}_A(M, N)$. Since G is finite, we have $\mathrm{HOM}_A(M, N) = \mathrm{Hom}_A(M, N)$ and $\mathrm{EXT}^i_A(M, N) = \mathrm{Ext}^i_A(M, N)$ (cf. [28, Corollary 2.4.6 and Corollary 2.4.7]). Now that $\mathrm{Ext}^i_A(M, N)$ may be computed from a graded projective resolution of M, the grading on each $\mathrm{Hom}_A(M, N)$ induces a grading on $\mathrm{Ext}^i_A(M, N)$. Therefore $\mathrm{Ext}^*_A(A/J, A/J)\#G^*$ is well-defined.

For any graded A-module M, $M \otimes_K KG^*$ is a left $A\#G^*$-module defined by $(a \otimes p_g)(m \otimes p_h) = am_{gh^{-1}} \otimes p_h$ for all $a \in A, m \in M$ and $g, h \in G$. If P is a finitely generated graded projective A-module then $P \otimes_K KG^*$ is a finitely generated projective $A\#G^*$-module.

3.2 Quasi-Koszulity of smash product

Lemma 9 *Let G be a finite group, A a G-graded K-algebra, P a finitely generated graded projective A-module and N a graded A-module. Then there is a K-vector space isomorphism $\theta_P : Hom_A(P, N) \otimes_K KG^* \to Hom_{A\#G^*}(P \otimes_K KG^*, N \otimes_K KG^*)$ defined by $\theta_P(\phi \otimes p_g)(x_h \otimes p_l) = \phi(x_h)_{hlg^{-1}} \otimes p_g$ for all $x_h \in P_h$ and $p_g, p_l \in KG^*$.*

Proof. Consider the natural isomorphism $\sigma : \mathrm{Hom}_A(A, N) \otimes_K KG^* \to N \otimes_K KG^*, \phi \otimes p_g \mapsto \phi(1) \otimes p_g$ and $\tau : \mathrm{Hom}(A \otimes_K KG^*, N \otimes_K KG^*) \to N \otimes_K KG^*, \psi \mapsto \psi(1 \otimes 1)$. We have $\theta_A = \tau^{-1}\sigma$. Thus Lemma 9 holds for $P = A$. Next we can prove routinely that Lemma 9 holds for all A^n with $n \geq 1$ and all finitely generated projective $A * G$-module P. □

Lemma 10 *Let G be a finite group, A a G-graded K-algebra, M a graded A-module admitting a finitely generated graded projective resolution $\mathbb{P}$ =*

$(P_i, \psi_i)_{i\geq 0}$ *and N a graded A-module. Then there is a natural isomorphism* $\overline{\theta}_i : Ext^i_A(M, N) \otimes_K KG^* \to Ext^i_{A\#G^*}(M \otimes_K KG^*, N \otimes_K KG^*)$ *defined by* $\overline{\theta}_i(\overline{\phi} \otimes p_g) := \overline{\theta_{P_i}(\phi \otimes p_g)}$ *for each* $i \geq 0$.

Proof. Note that $\mathbb{P} \otimes_K KG^* = (P_i \otimes_K KG^*, \psi_i \otimes 1)_{i\geq 0}$ is a finitely generated $A\#G^*$-module projective resolution of $M \otimes_K KG^*$. It follows from Lemma 9 that $(\theta_{P_i})_{i\geq 0}$ is a chain isomorphism from the complex $\mathrm{Hom}_A(\mathbb{P}, N)\otimes_K KG^*$ to the complex $\mathrm{Hom}_{A\#G^*}(\mathbb{P} \otimes_K KG^*, N \otimes_K KG^*)$. This chain isomorphism induces isomorphisms $\overline{\theta}_i : \mathrm{Ext}^i_A(M, N) \otimes_K KG^* \to \mathrm{Ext}^i_{A\#G^*}(M \otimes_K KG^*, N \otimes_K KG^*)$ between the homologies of these two complexes defined by $\overline{\theta}_i(\overline{\phi} \otimes p_g) := \overline{\theta_{P_i}(\phi \otimes p_g)}$ for all $i \geq 1$. The case $i = 0$ follows from Five Lemma. $\square$

Theorem 3 *Let A be a finite-dimensional G-graded K-algebra, where G is a finite group satisfying $charK \nmid |G|$. Let M be a finitely generated graded A-module. Then $M \otimes_K KG^*$ is a quasi-Koszul $A\#G^*$-module if and only if M is a quasi-Koszul A-module. In particular, $A\#G^*$ is quasi-Koszul if and only if so is A. Moreover,* $Ext^*_{A\#G^*}((A\#G^*)/J(A\#G^*), (A\#G^*)/J(A\#G^*)) \cong Ext^*_A(A/J, A/J)\#G^*$ *as positively graded K-algebras.*

Proof. Let $\mathbb{P} = (P_k, \psi_k)_{k\geq 0}$ be a finitely generated graded projective resolution of M. First of all, we show that the following diagram is commutative:

$$\begin{array}{ccc}
(\mathrm{Ext}^i_A(M, A/J) \otimes_K KG^*) \times (\mathrm{Ext}^j_A(A/J, A/J) \otimes_K KG^*) & \xrightarrow{\mu} & \mathrm{Ext}^{i+j}_A(M, A/J) \otimes_K KG^* \\
\downarrow (\bar{\theta}_i, \bar{\theta}_j) & & \downarrow \bar{\theta}_{i+j} \\
\mathrm{Ext}^i_{A\#G^*}(M \otimes_K KG^*, (A/J)\#G^*) \times \mathrm{Ext}^j_{A\#G^*}((A/J)\#G^*, (A/J)\#G^*) & \xrightarrow{\nu} & \mathrm{Ext}^{i+j}_{A\#G^*}(M \otimes_K KG^*, (A/J)\#G^*)
\end{array}$$

where ν is Yoneda product and μ is defined by $\mu(\bar{\alpha}\otimes p_g, \bar{\beta}\otimes p_h) := \bar{\alpha}(\bar{\beta})_{gh^{-1}}\otimes p_h$.

Let $\mathbb{Q} = (Q_k, \phi_k)_{k\geq 0}$ be a finitely generated graded projective resolution of A/J. Let $\bar{\alpha} \in \mathrm{Ext}^i_A(M, A/J)$ and $\bar{\beta} \in \mathrm{Ext}^j_A(A/J, A/J)$ with $\alpha \in \mathrm{Hom}_A(\Omega^i(M), A/J)$ or $\mathrm{Hom}(P_i, A/J)$ and $\beta \in \mathrm{Hom}_A(\Omega^j(A/J), A/J)$ or $\mathrm{Hom}(Q_j, A/J)$. Then $\bar{\alpha}(\bar{\beta})_{gh^{-1}} = \overline{\beta_{gh^{-1}}\Omega^j(\alpha)}$. Since $\mathbb{P} = (P_k, \psi_k)_{k\geq 0}$ is a graded projective resolution of M, $\mathbb{P}(i) = (P_{k+i}, \psi_{k+i})_{k\geq 0}$ is a graded projective resolution $\Omega^i(M)$ for all $i \geq 0$. Furthermore $\mathbb{P}(i) \otimes_K KG^* = (P_{k+i} \otimes_K KG^*, \psi_{k+i} \otimes 1)_{k\geq 0}$ is an $A\#G^*$-module projective resolution of $\Omega^i(M) \otimes_K KG^*$ for all $i \geq 0$. Suppose $\alpha : \Omega^i(M) \to A/J$ induces a chain map $(\alpha_j)_{j\geq 0} : \mathbb{P}(i) \to \mathbb{Q}$. Then $(\theta_{P_{j+i}}(\alpha_j \otimes p_g))_{j\geq 0} : \mathbb{P}(i) \otimes_K KG^* \to \mathbb{Q}\otimes_K KG^*$ is exactly a chain map induced by the map $\theta_{P_i}(\alpha\otimes p_g)$. Therefore $\theta_{P_{j+i}}(\Omega^j(\alpha) \otimes p_g) = \Omega^j(\theta_{P_i}(\alpha \otimes p_g))$.

For any $x \in (P_{j+i})_l$ and $p_m \in KG^*$, we have

$$\begin{aligned}
& \theta_{Q_j}(\beta \otimes p_h)\Omega^j(\theta_{P_i}(\alpha \otimes p_g))(x \otimes p_m) \\
= \ & \theta_{Q_j}(\beta \otimes p_h)\theta_{P_{j+i}}(\Omega^j(\alpha) \otimes p_g))(x \otimes p_m) \\
= \ & \theta_{Q_j}(\beta \otimes p_h)((\Omega^j(\alpha)(x))_{mlg^{-1}} \otimes p_g) \\
= \ & \beta((\Omega^j(\alpha)(x))_{mlg^{-1}})_{mlh^{-1}} \otimes p_h \\
= \ & (\beta_{gh^{-1}}\Omega^j(\alpha)(x))_{mlh^{-1}} \otimes p_h \\
= \ & \theta_{P_{j+i}}(\beta_{gh^{-1}}\Omega^j(\alpha) \otimes p_h)(x \otimes p_m),
\end{aligned}$$

Thus $\bar{\theta}_{i+j}((\bar{\alpha} \otimes p_g)(\bar{\beta} \otimes p_h)) = \bar{\theta}_{i+j}(\bar{\alpha}(\bar{\beta})_{g^{-1}h} \otimes p_h) = \bar{\theta}_{i+j}(\overline{\beta_{gh^{-1}}\Omega^j(\alpha)} \otimes p_h) = \overline{\theta_{P_{j+i}}(\beta_{gh^{-1}}\Omega^j(\alpha) \otimes p_h)} = \overline{\theta_{Q_j}(\beta \otimes p_h)\Omega^j(\theta_{P_i}(\alpha \otimes p_g))} = \overline{\theta_{P_i}(\alpha \otimes p_g)}\,\overline{\theta_{Q_j}(\beta \otimes p_h)} = \bar{\theta}_i(\bar{\alpha} \otimes p_g)\bar{\theta}_j(\bar{\beta} \otimes p_h)$. So the diagram above is commutative.

Let J and J_G be the Jacobson radical and graded Jacobson radical of A respectively. Since char$K \nmid |G|$, by [11, Theorem 4.1 and Theorem 4.4], we have $J(A\#G^*) = J_G\#G^* = J\#G^*$. Moreover, $(A\#G^*)/J(A\#G^*) \cong (A/J)\#G^*$.

Therefore $M \otimes_K KG^*$ is a quasi-Koszul $A\#G^*$-module if and only if

$$\begin{aligned}
& \mathrm{Ext}^{i+j}_{A\#G^*}(M \otimes_K KG^*, (A/J)\#G^*) \\
= \ & \mathrm{Ext}^{i}_{A\#G^*}(M \otimes_K KG^*, (A/J)\#G^*)\mathrm{Ext}^{j}_{A\#G^*}((A/J)\#G^*, (A/J)\#G^*)
\end{aligned}$$

for all $i, j \geq 0$, if and only if

$$\begin{aligned}
& \mathrm{Ext}^{i+j}_{A}(M, A/J) \otimes_K KG^* \\
= \ & (\mathrm{Ext}^{i}_{A}(M, A/J) \otimes_K KG^*)(\mathrm{Ext}^{j}_{A}(A/J, A/J) \otimes_K KG^*) \\
= \ & (\mathrm{Ext}^{i}_{A}(M, A/J)\mathrm{Ext}^{j}_{A}(A/J, A/J)) \otimes_K KG^*
\end{aligned}$$

for all $i, j \geq 0$, if and only if

$$\mathrm{Ext}^{i+j}_{A}(M, A/J) = \mathrm{Ext}^{i}_{A}(M, A/J)\mathrm{Ext}^{j}_{A}(A/J, A/J)$$

for all $i, j \geq 0$, if and only if M is a quasi-Koszul A-module. In particular, taking $M = A/J$ we have the algebra $A\#G^*$ is quasi-Koszul if and only if so is A. Moreover, according to the commutative diagram above, the map $(\bar{\theta}_i)_{i\geq 0}$ is a positively graded K-algebra isomorphism $\mathrm{Ext}^*_A(A/J, A/J)\#G^* \cong \mathrm{Ext}^*_{A\#G^*}((A\#G^*)/J(A\#G^*), (A\#G^*)/J(A\#G^*))$. $\square$

Remark 3 *Let A be a finite-dimensional G-graded K-algebra, where G is a finite group satisfying char$K \nmid |G|$. We can show in another way that A is quasi-Koszul if and only if so is $A\#G^*$: It follows from Corollary 1, Cohen-Montgomery's duality theorem for coaction (cf. [11, Theorem 3.5]) and Theorem 2 that A is quasi-Koszul if and only if so is $M_n(A) = (A\#G^*) * G$, if and only if so is $A\#G^*$.*

4 Finite Galois covering and Koszulity

In this section, we show that if a finite-dimensional connected quiver algebra is Koszul then so are its Galois covering algebras with finite Galois group G satisfying $\mathrm{char} K \nmid |G|$. Indeed, we have two ways to observe the relations between finite Galois covering and (quasi-)Koszulity: One is via covering functor and galois extension (cf. Section 4.1), the other is via covering of quiver with relations (cf. Section 4.2).

4.1 Galois extension, covering functor and Koszulity

Let K be an algebraically closed field. Let A' be a K-algebra and G a finite group acting on A' as a group of K-automorphisms. We say that the pair (A', G) is *pregalois* if A' is a projective $A' {*} G$-generator (cf. [2, Section 1]). We say that the pregalois pair (A', G) is *left* (resp. *right*) *galois* if $A'^G/\mathrm{ann}_{A'^G} S$ is a semisimple artinian K-algebra for each simple left (resp. right) A'-module S. We say that the pair (A', G) is *galois* if it is both left and right galois (cf. [2, Section 4]). In this case, the K-algebra A' is called a *galois extension* of $A := A'^G$. In case A' is a finite-dimensional basic K-algebra, (A', G) is galois if and only if the induced action of G on the isomorphism classes of simple A'-modules is free (cf. [2, Proposition 5.3]).

A *K-category* $\underline{A}$ is a preadditive category in which the morphism sets are K-vector spaces and the compositions are K-bilinear. A K-category $\underline{A}$ is called a *locally bounded K-category* if it satisfies: (1) for each object a in $\underline{A}$, $\mathrm{End}_{\underline{A}}(a)$ is a local ring; (2) for each pair of objects a and b in $\underline{A}$, $\dim_K \mathrm{Hom}_{\underline{A}}(a, b) < \infty$; (3) distinct objects of $\underline{A}$ are nonisomorphic; and (4) for each a in $\underline{A}$ there are only a finite number of objects b in $\underline{A}$ such that $\mathrm{Hom}_{\underline{A}}(a, b) \neq 0$ or $\mathrm{Hom}_{\underline{A}}(b, a) \neq 0$ (cf. [9, Section 2.1]).

If $\underline{A}'$ and $\underline{A}$ are locally bounded K-categories then a K-linear functor $\underline{F} : \underline{A}' \to \underline{A}$ is called a *covering functor* if: (1) $\underline{F}$ is surjective on objects; and (2) for each a' in $\underline{A}'$ and a in $\underline{A}$, $\underline{F}$ induces isomorphisms $\coprod_{b' \in \underline{F}^{-1}(a)} \mathrm{Hom}_{\underline{A}'}(b', a') \to \mathrm{Hom}_{\underline{A}}(a, \underline{F}(a'))$ and $\coprod_{b' \in \underline{F}^{-1}(a)} \mathrm{Hom}_{\underline{A}'}(a', b') \to \mathrm{Hom}_{\underline{A}}(\underline{F}(a'), a)$ (cf. [9, Section 3.1]).

Let $\underline{A}'$ be a finite locally bounded K-category. Assume that G is a finite group of K-automorphisms of $\underline{A}'$ such that the induced action on the objects is free. Then, by [13, 3.1 Proposition], the quotient category $\underline{A}'/G$ exists and the canonical projection $\underline{A}' \to \underline{A}'/G$ is a covering functor, which is called a *Galois covering with finite Galois group G*.

Let A be a finite-dimensional basic K-algebra. Then A is isomorphic to KQ/I where Q is a finite quiver and I is an admissible ideal of the path algebra KQ (cf. [3, Corollary 1.10]). So A can be viewed as a finite locally

bounded K-category $\underline{A}$ which is the quotient category of the path category KQ (cf. [9, Section 2.1]).

Assume that $\underline{F} : \underline{A}' \to \underline{A}$ is a Galois covering with finite Galois group G, where the finite locally bounded K-category $\underline{A}'$ corresponds to a finite-dimensional basic K-algebra $A' = KQ'/I'$. Then (A', G) is a galois extension and $A = A'^G$ (cf. [2, Theorem 6.2]). It follows from Theorem 2 that, in the case of char$K \nmid |G|$, A' is quasi-Koszul if and only if so is $A' * G$. By [2, Propositon 1.2], we know the category Mod$A' * G$ is equivalent to the category ModA'^G, i.e., Mod$A$. Applying Theorem 1 we have $A' * G$ is quasi-Koszul if and only if so is $A = A'^G$. Thus, in the case of char$K \nmid |G|$, A' is quasi-Koszul if and only if so is A.

If both $A = KQ/I$ and $A' = KQ'/I'$ are graded quiver algebras then A is Koszul if and only if so is A'.

Remark 4 *Though we can obtain the relation between Koszulity and finite Galois covering via galois extension, the underlying field K has to be assumed to be algebraically closed.*

4.2 Covering of quiver with relations and Koszulity

Let K be a field and Q a quiver. From now on we always denote by Q_0 (resp. Q_1) the vertex (resp. arrow) set of Q. For a path p in Q, we denote by $i(p)$ (resp. $t(p)$) the initial point (resp. terminal) point of p. Let Q' and Q be quivers. Let $F : Q' \to Q$ be a covering (in the topological sense) and $v' \in Q'_0$. Let $\pi_1(Q', v')$ be the fundamental group of Q' and $F_* : \pi_1(Q', v') \to \pi_1(Q, F(v'))$ be the map induced by F. We say a covering $F : Q' \to Q$ is *regular* if $F_*(\pi_1(Q', v'))$ is a normal subgroup of $\pi_1(Q, F(v'))$. In this case, Q is homeomorphic to the quotient space Q'/G where $G \cong \pi_1(Q, F(v'))/F_*(\pi_1(Q', v'))$ is the automorphism group of the covering $F : Q' \to Q$ (cf. [26, Chapter 5]).

Some of the following contents are taken from [14] for reader's convenience. If $r = \sum_{i=1}^m k_i p_i$ is a K-linear combination of paths in Q and v, w are vertices in Q, we denote by $c_{v,w}(r)$ the (v, w)-*component of* r where $c_{v,w}(r) := \sum_{j=1}^n k_{i_j} p_{i_j}$ and $\{p_{i_j}\}_{j=1}^n$ is the subset of $\{p_i\}_{i=1}^m$ of all those paths with initial point v and terminal point w.

A map $L : Q_0 \to Q'_0$ is called a *lifting* if $L(v) \in F^{-1}(v)$ for all $v \in Q_0$. By uniqueness of path lifting, if p is a path with initial point $i(p)$ and terminal point $t(p)$ then we denote by $L(p)$ the unique path in Q' with initial point $L(i(p))$ such that $F(L(p)) = p$. If $r = \sum_{i=1}^m k_i p_i$ is a K-linear combination of paths in Q we denote $\sum_{i=1}^m k_i L(p_i)$ by $L(r)$.

We say (Q, ρ) is a *quiver with relations* if Q is a quiver and ρ is a set of K-linear combinations of paths in Q. Let (Q', ρ') and (Q, ρ) be quivers

with relations. We say $F : (Q', \rho') \to (Q, \rho)$ is a *morphism of quivers with relations* if $F : Q' \to Q$ is a regular covering of quivers satisfying: (1) $\rho' = \{L(r) | L : Q_0 \to Q'_0 \text{ is a lifting and } r \in \rho\}$; (2) if $r' \in \rho'$ and $v, w \in Q_0$ then there exist $v', w' \in Q'_0$ such that $F(c_{v',w'}(r')) = c_{v,w}(F(r'))$.

Let $A = KQ/\langle\rho\rangle$ be a finite-dimensional connected K-algebra where $\langle\rho\rangle$ denotes the admissible ideal of KQ generated by ρ. Let $F : (Q', \rho') \to (Q, \rho)$ be a morphism of quivers with relations and finite group G its group of automorphisms. In this case, F is called a *Galois covering with finite Galois group* G. It follows from [14, Theorem 3.2]) that there is a weight function $W : Q_1 \to G$ such that A may be given the G-grading induced by W. Moreover, $\text{mod}KQ'/\langle\rho'\rangle$ is equivalent to the category $\text{gr}A$ of finite-dimensional G-graded A-modules. By [11, Theorem 2.2], we have $\text{gr}A$ is isomorphic to $\text{mod}A\#G^*$. Thus $\text{mod}KQ'/\langle\rho'\rangle$ is equivalent to $\text{mod}A\#G^*$. It follows from Theorem 1 and Theorem 3 that, in the case of $\text{char}K \nmid |G|$, $KQ'/\langle\rho'\rangle$ is quasi-Koszul if and only if so is $A\#G^*$, if and only if so is A.

Clearly, ρ is a set of homogeneous relations in the length grading if and only if so is ρ'. Thus, if $KQ/\langle\rho\rangle$ is a graded quiver algebra with ρ a set of homogeneous relations in the length grading then so is $A' = KQ'/\langle\rho'\rangle$. So we obtain the following theorem:

Theorem 4 *Let $KQ/\langle\rho\rangle$ be a finite-dimensional connected Koszul K-algebra with ρ a set of quadratic relations. Let $F : (Q', \rho') \to (Q, \rho)$ be a Galois covering with finite Galois group G satisfying char$K \nmid |G|$. Then $KQ'/\langle\rho'\rangle$ is Koszul.*

Remark 5 *Of course we also have the "inverse" of Theorem 4 as follows: Let $KQ'/\langle\rho'\rangle$ be a finite-dimensional connected Koszul K-algebra with ρ' a set of quadratic relations. Let $F : (Q', \rho') \to (Q, \rho)$ be a Galois covering with finite Galois group G satisfying char$K \nmid |G|$. Then $KQ/\langle\rho\rangle$ is Koszul. However, it is useless in practice.*

5 Construction of Koszul algebras

In this section, we provide a general construction of Koszul algebra by Galois covering with finite cyclic Galois group. Moreover, as examples, we construct Koszul algebras from exterior algebras and Koszul preprojective algebras by finite Galois covering with either cyclic or noncyclic Galois group.

5.1 General construction

Let $A = KQ/\langle\rho\rangle$ be a Koszul algebra with ρ a set of quadratic relations. Then one can construct a finite Galois covering $F : (Q', \rho') \to (Q, \rho)$ with Galois group $G = \mathbb{Z}_n, n \geq 2$, satisfying $\mathrm{char}K \nmid n$, in the following way: Let $Q'_0 := \{(v, \overline{r}) | v \in Q_0, \overline{r} \in \mathbb{Z}_n\}$, $Q'_1 := \{(a, \overline{r}) : (i(a), \overline{r}) \to (t(a), \overline{r+1}) | a \in Q_1, r \in \mathbb{Z}\}$ and $\rho' = \{\sum_{i=1}^{s} k_i(a_{i2}, \overline{r+2})(a_{i1}, \overline{r+1}) | \sum_{i=1}^{s} k_i a_{i2} a_{i1} \in \rho, k_i \in K, a_{ij} \in Q_1, r \in \mathbb{Z}\}$. The covering $F : (Q', \rho') \to (Q, \rho)$ is defined by $(v, \overline{r}) \mapsto v$ and $(a, \overline{r}) \mapsto a$, which is a Galois covering with Galois group $\mathbb{Z}_n$. By Theorem 4, the graded quiver algebras $KQ'/\langle\rho'\rangle$ are Koszul for all $n \geq 2$ with $\mathrm{char}K \nmid n$.

5.2 Constructions from exterior algebras

Exterior algebras is a quite important class of algebras, which plays extremely important roles in many mathematical branches such as algebraic geometry, commutative algebra, differential geometry.

Let Q be the quiver given by one vertex 1 and m-loops $a_1, a_2, ..., a_m$ with $m \geq 2$. Denote by ρ the set $\{a_i^2 | 1 \leq i \leq m\} \cup \{a_i a_j + a_j a_i | 1 \leq i < j \leq m\}$. Then $A := KQ/\langle\rho\rangle$ is the exterior algebra over K (cf. [27]). It is well-known that A is a Koszul algebra and its quadratic dual is the algebra $K[x_1, ..., x_m]$ of polynomials in m variables $x_1, ..., x_m$.

Applying the general construction in Section 5.1, we can obtain many new Koszul algebras:

Example 1 *Define the quiver with relations* (Q', ρ') *by* $Q'_0 := \{\overline{r} | \overline{r} \in \mathbb{Z}_n\}$, $Q'_1 := \{(a_i, \overline{r}) : \overline{r} \to \overline{r+1} | 1 \leq i \leq m, r \in \mathbb{Z}\}$ *and* $\rho' = \{(a_i, \overline{r+1})(a_i, \overline{r}) | 1 \leq i \leq m, r \in \mathbb{Z}\} \cup \{(a_i, \overline{r+1})(a_j, \overline{r}) + (a_j, \overline{r+1})(a_i, \overline{r}) | 1 \leq i < j \leq m, r \in \mathbb{Z}\}$. *The covering* $F : (Q', \rho') \to (Q, \rho)$ *is defined by* $\overline{r} \mapsto 1$ *and* $(a, \overline{r}) \mapsto a$, *which is a Galois covering with Galois group* $\mathbb{Z}_n$. *By Theorem 4, the graded quiver algebras* $KQ'/\langle\rho'\rangle$ *are Koszul for all* $n \geq 2$ *with* $\mathrm{char}K \nmid n$.

We can construct more new Koszul algebras from exterior algebras by finite Galois covering with cyclic Galois group:

Example 2 *For* $1 \leq l \leq m$, *define the quiver with relations* (Q', ρ') *by* $Q'_0 := \{\overline{r} | \overline{r} \in \mathbb{Z}_n\}$, $Q'_1 := \{(a_i, \overline{r}) : \overline{r} \to \overline{r+1} | 1 \leq i \leq l, r \in \mathbb{Z}\} \cup \{(a_i, \overline{r}) : \overline{r+1} \to \overline{r} | l+1 \leq i \leq m, r \in \mathbb{Z}\}$ *and* ρ' *naturally induced by* ρ, *i.e.,* $\rho' := \{(a_i, \overline{r+1})(a_i, \overline{r}) | 1 \leq i \leq l, r \in \mathbb{Z}\} \cup \{(a_i, \overline{r})(a_i, \overline{r+1}) | l+1 \leq i \leq m, r \in \mathbb{Z}\} \cup \{(a_j, \overline{r+1})(a_i, \overline{r}) + (a_i, \overline{r+1})(a_j, \overline{r}) | 1 \leq i < j \leq l, r \in \mathbb{Z}\} \cup \{(a_j, \overline{r})(a_i, \overline{r+1}) + (a_i, \overline{r})(a_j, \overline{r+1}) | l+1 \leq i < j \leq m, r \in \mathbb{Z}\} \cup \{(a_j, \overline{r})(a_i, \overline{r}) + (a_i, \overline{r-1})(a_j, \overline{r-1}) | 1 \leq i \leq l, l+1 \leq j \leq m, r \in \mathbb{Z}\}$. *The*

covering $F : (Q', \rho') \to (Q, \rho)$ is defined by $\overline{r} \mapsto 1$ and $(a_i, \overline{r}) \mapsto a_i$, which is a Galois covering with Galois group $\mathbb{Z}_n$. By Theorem 4, the graded quiver algebras $KQ'/\langle \rho' \rangle$ are Koszul for all $n \geq 2$ with char$K \nmid n$.

Remark 6 *It is well-known that the exterior algebras can be viewed as $\mathbb{Z}_2$-graded algebras. The graded exterior algebras (called Grassmann algebras as well [12]) have applications in physics. In the case of $n = 2$, namely in the case the Galois group is $\mathbb{Z}_2$, by [14, Theorem 3.2], the category mod$KQ'/\langle \rho' \rangle$ is equivalent to the category gr$KQ/\langle \rho \rangle$ of $\mathbb{Z}_2$-graded modules (or superrepresentations) over the graded exterior algebra (or superalgebra) $KQ/\langle \rho \rangle$. From this viewpoint, the algebras $KQ'/\langle \rho' \rangle$ in Example 1 and Example 2 should be important.*

We can also construct new Koszul algebras from exterior algebras by finite Galois covering with non-cyclic Galois group:

Example 3 *Consider the exterior algebra with $m = 2$. Define the quiver with relations (Q', ρ') by $Q'_0 := \{(1, \overline{r}) | \overline{r} \in \mathbb{Z}_n\} \cup \{(1', \overline{r}) | \overline{r} \in \mathbb{Z}_n\}$, $Q'_1 := \{(a_1, \overline{r}) : (1, \overline{r}) \to (1', \overline{r}) | r \in \mathbb{Z}\} \cup \{(a'_1, \overline{r}) : (1', \overline{r}) \to (1, \overline{r}) | r \in \mathbb{Z}\} \cup \{(a_2, \overline{r}) : (1, \overline{r}) \to (1, \overline{r+1}) | r \in \mathbb{Z}\} \cup \{(a'_2, \overline{r}) : (1', \overline{r}) \to (1', \overline{r+1}) | r \in \mathbb{Z}\}$ and ρ' naturally induced by $\rho = \{a_1^2, a_2^2, a_1 a_2 + a_2 a_1\}$. The covering $F : (Q', \rho') \to (Q, \rho)$ is defined by $(1, \overline{r}) \mapsto 1$, $(1', \overline{r}) \mapsto 1$, $(a_i, \overline{r}) \mapsto a_i$ and $(a'_i, \overline{r}) \mapsto a_i$, which is a Galois covering with Galois group $\mathbb{Z}_2 \times \mathbb{Z}_n$. By Theorem 4, the graded quiver algebras $KQ'/\langle \rho' \rangle$ are Koszul for all $n \geq 2$ with char$K \nmid 2n$.*

We can also construct new Koszul algebras from exterior algebras by finite Galois covering with Galois group not the direct product of cyclic groups:

Example 4 *Consider the exterior algebra with $m = 2$. Define the quiver with relations (Q', ρ') by $Q'_0 := \{(1, \overline{r}) | \overline{r} \in \mathbb{Z}_n\} \cup \{(1', \overline{r}) | \overline{r} \in \mathbb{Z}_n\}$, $Q'_1 := \{(a_1, \overline{r}) : (1, \overline{r}) \to (1', \overline{r}) | r \in \mathbb{Z}\} \cup \{(a'_1, \overline{r}) : (1', \overline{r}) \to (1, \overline{r}) | r \in \mathbb{Z}\} \cup \{(a_2, \overline{r}) : (1, \overline{r}) \to (1, \overline{r+1}) | r \in \mathbb{Z}\} \cup \{(a'_2, \overline{r}) : (1', \overline{r+1}) \to (1', \overline{r}) | r \in \mathbb{Z}\}$ and ρ' naturally induced by $\rho = \{a_1^2, a_2^2, a_1 a_2 + a_2 a_1\}$. The covering $F : (Q', \rho') \to (Q, \rho)$ is defined by $(1, \overline{r}) \mapsto 1$, $(1', \overline{r}) \mapsto 1$, $(a_i, \overline{r}) \mapsto a_i$ and $(a'_i, \overline{r}) \mapsto a_i$, which is a Galois covering with Galois group the dihedral group D_{2n}. By Theorem 4, the graded quiver algebras $KQ'/\langle \rho' \rangle$ are Koszul for all $n \geq 2$ with char$K \nmid 2n$.*

5.3 Constructions from Preprojective algebras

Preprojective algebras is a very nice class of algebras, which plays important roles in differential geometry and quantum groups.

Let $Q = (Q_0, Q_1)$ be a finite connected quiver without oriented cycle. Then the double quiver $\overline{Q}$ is defined by $\overline{Q}_0 := Q_0$ and $\overline{Q}_1 := Q_1 \cup \{a^* : t(a) \to i(a) | a \in Q_1\}$. Suppose that $\overline{I}$ is the ideal of $K\overline{Q}$ generated by all elements of the form $\sum_{a \in Q_1}(aa^* - a^*a)$. Then the algebra $\mathcal{P}(Q) := K\overline{Q}/\overline{I}$ is called the *preprojective algebras of the quiver* Q (cf. [32]).

Note that $\mathcal{P}(Q)$ is Koszul if and only if Q is either $\mathbb{A}_1, \mathbb{A}_2$ or is not a Dynkin diagram [23, Theorem 1.9]. In the case of $Q = \mathbb{A}_1$, $\mathcal{P}(\mathbb{A}_1) = K$. In the case of $Q = \mathbb{A}_2$, the algebra $\mathcal{P}(\mathbb{A}_2)$ is radical square zero, so is its finite Galois covering algebras. Thus we cannot construct new Koszul algebras from $\mathcal{P}(\mathbb{A}_1)$ and $\mathcal{P}(\mathbb{A}_2)$ using our approach. Since $\mathcal{P}(Q)$ is finite-dimensional if and only if Q is a Dynkin quiver, our approach cannot be applied to preprojective algebras directly. However, we can construct many new Koszul algebras from the quadratic dual of preprojective algebras, which are finite-dimensional.

Let K be an algebraically closed field. It follows from [23, Theorem 1.9] that, in case Q is a tree but not a Dynkin diagram, the quadratic dual of $\mathcal{P}(Q)$ is just the trivial extension $\mathcal{D}(Q)$ of the path algebra $K\tilde{Q}$, where $\tilde{Q}$ is of the same underlying graph as Q and each vertex is either a source or a sink. By [23, Theorem 1.8], $\mathcal{D}(Q) \cong K\hat{Q}/\hat{I}$ where $\hat{Q}$ is defined by $\hat{Q}_0 := Q_0$ and $\hat{Q}_1 := \overline{Q}_1$ and the ideal $\hat{I}$ is generated by the set $\hat{\rho}$ which consists of the following relations: (1) $a^*a - b^*b$, if $i(a) = i(b)$ is a source in $\hat{Q}$; (2) $aa^* - bb^*$, if $t(a) = t(b)$ is a sink in $\hat{Q}$; (3) b^*a, if $t(a) = t(b)$ and $a \neq b$; (4) ab^*, if $i(a) = i(b)$ and $a \neq b$.

On one hand, applying the general construction in Section 5.1 to $K\hat{Q}/\langle\hat{\rho}\rangle$, we can construct many new Koszul algebras. On the other hand, we can also construct more new Koszul algebras in the following way:

Example 5 *Define the quiver with relations* (Q', ρ') *by* $Q'_0 := \{(v, \overline{r}) | v \in Q_0, \overline{r} \in \mathbb{Z}_n\}$, $Q'_1 := \{(a, \overline{r}) : (i(a), \overline{r}) \to (t(a), \overline{r}) | a \in Q_1, \overline{r} \in \mathbb{Z}_n\} \cup \{(a^*, \overline{r}) : (i(a^*), \overline{r}) \to (t(a^*), \overline{r+1}) | a \in Q_1, r \in \mathbb{Z}\}$ *and* ρ' *naturally induced by* $\hat{\rho}$. *The covering* $F : (Q', \rho') \to (\hat{Q}, \hat{\rho})$ *defined by* $(v, \overline{r}) \mapsto v$, $(a, \overline{r}) \mapsto a$ *and* $(a^*, \overline{r}) \mapsto a^*$, *is a Galois covering with Galois group* $\mathbb{Z}_n$. *By Theorem 4, the graded quiver algebras* $KQ'/\langle\rho'\rangle$ *are Koszul for all* $n \geq 2$ *with* $\mathrm{char}K \nmid n$.

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